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Analytic angle factors for the radiant interchange among the surface elements of two concentric cylinders

H. BROCKMANN

Institut für Sicherheitsforschung und Reaktortechnik, Forschungszentrum Jülich GmbH, 52425 Jülich, FRG

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Abstract—In the theory of thermal radiation the radiant interchange between diffusely emitting surfaces is described by the use of angle factors. In this study the angle factors for the radiation transfer between the surface elements of two concentric cylinders are calculated analytically. This is achieved by replacing one surface integration in the equation for the angle factor by an integration over the direction of the radiation emitted from that surface. One result of the study is that analytical angle factors are given for geometrical transitions for which apparently no information is available in the literature. Furthermore, simplified expressions for the known angle factors are derived and tabulated in a consistent notation facilitating their use in practical applications. The angle factors thus obtained can be easily evaluated numerically and used in heat transfer or neutral particle streaming calculations in r–z geometry.

1. INTRODUCTION

THE CONCEPT of angle factors (alternatively designated as view factors, geometry factors, configuration factors, or shape factors) has been developed in the theory of thermal radiation to compute the radiant interchange between diffusely emitting surfaces [1]. In the past few years, angle factors have also been used successfully in combination with diffusion and transport codes to calculate the neutral particle streaming (neutrons or photons) through large voids surrounded by material regions [2–4]. By applying these hybrid techniques the problems can be circumvented which may arise when the common diffusion or transport codes are used to calculate the neutron or photon flux distributions in configurations with large voids. Thus, the diffusion theory fails in voids and the discrete ordinates method, which is mainly used for solving the transport equation, shows the problem of ray effects which may occur if there are isolated sources at the surface of the void and the void is strongly elongated in one direction [5]. The idea of the hybrid techniques is then to use the proved diffusion or discrete ordinates methods for calculating the radiation transport in the material regions and the angle factors for treating the radiation exchange between the surface elements of the void regions.

In practical applications the systems considered may exhibit many spatial meshes and thus area elements at the void surface. In such situations the evaluation of a large number of angle factors is required, which may amount to several thousands. This is a time-consuming process if the angle factors are calculated numerically and adequate accuracy is to be achieved. For this reason it is advantageous to have analytical expressions for the angle factors in these cases.

Based on the investigations on this subject described in ref. [4] the analytical angle factors for the radiant interchange among the surface elements of two concentric cylinders are calculated in a systematic manner. One result is that analytical angle factors are obtained for geometrical transitions for which apparently no complete solutions are given in the literature. Furthermore, simplified expressions with a consistent notation are derived for the known angle factors facilitating their use in practice. In this paper the procedure for calculating the analytical angle factors is described and the resulting expressions are compiled considering limiting conditions as well.

2. CALCULATION OF THE ANGLE FACTORS

The angle factor for the radiant interchange between two finite surfaces $F_{k'}$ and F_k is defined by

$$t_{k'k} = \frac{1}{\pi F_{k'}} \int_{F_k} \mathrm{d}f \int_{F_{k'}} \mathrm{d}f' \frac{(\mathbf{n}\cdot\boldsymbol{\Omega})(\mathbf{n}'\cdot\boldsymbol{\Omega})}{|\mathbf{r}-\mathbf{r}'|^2} \quad (1)$$

where $\mathbf{r}$ is a point on the surface F_k, $\mathrm{d}f$ an infinitesimal surface element, and $\mathbf{n}$ a unit vector normal to the surface. The primed quantities are defined in the same manner for the surface $F_{k'}$. Furthermore, the vector $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \quad (2)$$

The angle factors for a given geometrical system are

NOMENCLATURE

df	infinitesimal surface element
$\mathbf{e}$	unit vector
F	surface area
H	cylinder height
$\mathbf{n}$	unit normal vector
r	radial coordinate
$\mathbf{r}$	position vector
R	cylinder or circle radius
t	angle factor
X	dimensionless variable
Y	dimensionless variable
z	axial coordinate.

Greek symbols

γ	constant
ε	constant
η	cosine of polar angle
ϕ	polar angle
ω	azimuthal angle
$\mathbf{\Omega}$	angular direction.

Subscripts

B	bottom
I	inner
k	incident
k′	outgoing
O	outer
T	top.

not independent of each other. Thus the reciprocity rule

$$F_{k'}t_{k'k} = F_k t_{kk'} \tag{3}$$

holds, which follows immediately from the definition of the angle factor. Furthermore, due to energy and particle conservation the enclosure rule

$$\sum_{k} t_{k'k} = 1 \tag{4}$$

must be fulfilled for a closed surface. In addition to the ordinary reciprocity rule there is an extended reciprocity rule which is valid for parallel or adjacent surfaces [1, 6].

In order to solve the integral in equation (1) it is advantageous to replace the integration over the surface $F_{k'}$ by an integration over the angular variable $\mathbf{\Omega}$. The solid angle element extended by df' at the space point $\mathbf{r}$ is given by

$$d\mathbf{\Omega} = \frac{\mathbf{n}'\cdot\mathbf{\Omega}\, df'}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{5}$$

The angle factor can thus be written in the form

$$t_{k'k} = \frac{1}{\pi F_{k'}}\int_{F_k} df \int_{\Delta\Omega_k} d\mathbf{\Omega}\, \mathbf{n}\cdot\mathbf{\Omega}. \tag{6}$$

where $\Delta\mathbf{\Omega}_{k'}$ denotes the angular range into which radiation is emitted from $F_{k'}$, which reaches the surface element df at $\mathbf{r}$.

Due to the geometry of the studied assembly, cylindrical coordinates r, ϕ, z are used for the spatial variable. The coordinate system is shown in Fig. 1. Since the radiant flux leaving a surface is assumed to be independent of the angle ϕ, the integration with respect to ϕ can be performed and only one spatial integration over r or z remains. It is furthermore convenient to give the coordinates of the direction vector $\mathbf{\Omega}$ in a coordinate system moving with the position vector $\mathbf{r}$, as shown in Fig. 1. One axis of the coordinate system is chosen parallel to the z-axis of the spatial coordinate system. The other axis points in the direction of the projection of $\mathbf{r}$ into the xy-plane. Using a polar coordinate system the coordinates of $\mathbf{\Omega}$ are given by (ω, η) where η is the cosine of the polar angle and ω the angle between the planes formed by the $\mathbf{\Omega}$ and $\mathbf{e}_z$ vectors and by the $\mathbf{e}_r$ and $\mathbf{e}_z$ vectors. The solid angle element can then be expressed by

$$d\mathbf{\Omega} = d\omega\, d\eta. \tag{7}$$

The components of the direction vector $\mathbf{\Omega}$ along the unit normal $\mathbf{n}$ on the cylinder surfaces are given by

$$\mathbf{n}\cdot\mathbf{\Omega} = \begin{cases} \sqrt{(1-\eta^2)}\cos\omega & \text{if } r = \text{const} \\ \eta & \text{if } z = \text{const.} \end{cases} \tag{8}$$

The ranges allowed for ω and η are determined by the geometrical bounds of the cylinder surfaces. These can be calculated as a function of the position vector $\mathbf{r}$ using the characteristic equations which relate the coordinates on the emitting surface to those on the

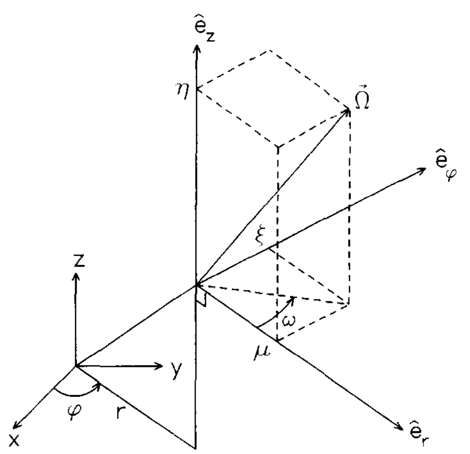

FIG. 1. Coordinates in cylinder geometry.

receiving surface. The characteristic equations may be written in the form

$$r\sin\omega = r'\sin\omega' \tag{9}$$

$$r\cos\omega - r'\cos\omega' = \gamma(z-z') \tag{10}$$

where γ is defined by

$$\gamma = \frac{\sqrt{(1-\eta^2)}}{\eta}. \tag{11}$$

If the radiation is interpreted as particle transport, the characteristic equations describe the particle trajectories during motion through a void region. They are solutions of the equation of motion using the local r–z–ω coordinate system described above. The geometrical meaning of the characteristic equations is illustrated in Fig. 2. While the azimuthal angle changes during particle motion, the value of η remains constant. The quantity γ, which represents the tangent of the polar angle, is therefore also constant. Thus, considering the projection of the motion into the xy-plane, equation (9) gives the distance of the particle trajectory from the origin of the coordinate system and equation (10) the distance of two points on the trajectory.

The η value for a transition from a point on the emitting surface to a point on the receiving surface is given by

$$\eta = \frac{z-z'}{\sqrt{((z-z')^2+(r\cos\omega - r'\cos\omega')^2)}} \tag{12}$$

or using equation (9) by

$$\eta = \frac{z-z'}{\sqrt{((z-z')^2+(r\cos\omega - \varepsilon\sqrt{(r'^2-r^2\sin^2\omega)})^2)}} \tag{13}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if } 0 \leqslant \omega' < \frac{\pi}{2} \\ -1 & \text{if } \frac{\pi}{2} < \omega' \leqslant \pi. \end{cases} \tag{14}$$

The integrations in equation (6) are first performed with respect to η and ω for a given value of r or z. While the integration limits of ω are given in an easy way by the bounds of the cylinder surfaces, the integration limits of η are calculated by the use of equation (13). Depending on the geometrical transitions either the difference $z-z'$ is constant and equal to the height of the cylinders or z' is constant and equal to the cylinder height. Furthermore, r' is constant and equal to the radius of the outer cylinder surface or of the top or bottom circular area. The subsequent integration over r or z gives the final solution for the angle factor.

For $\eta < 0$ there are eight geometrical transitions between the surfaces of the two concentric cylinders:

1. outer → outer
2. outer → inner
3. top → outer
4. top → bottom
5. top → inner
6. inner → outer
7. outer → bottom
8. inner → bottom.

The different situations are schematically shown in Fig. 3. The angle factors for the last three transitions can be calculated from the other transitions using the reciprocity rules. The angle factor for the transition top → inner can be determined from the angle factors for the transitions top → outer and top → bottom and by the use of the enclosure property. Thus, the angle factors for all transitions can be computed from those for the first four transitions. The angle factors for $\eta > 0$ can be treated in a similar way.

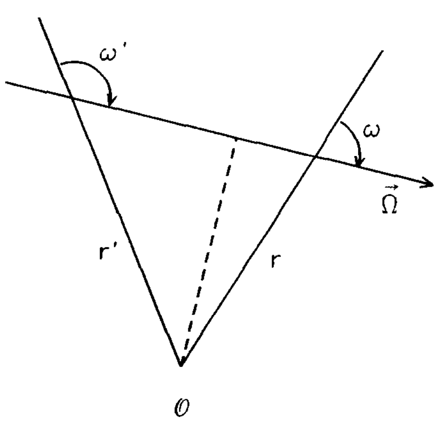

FIG. 2. Geometric illustration of the characteristic equations.

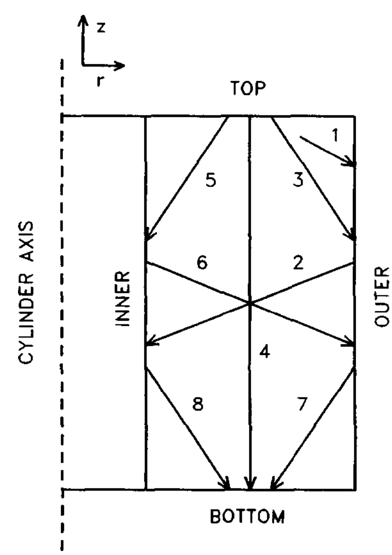

FIG. 3. The downward directed transitions between the surface elements.

3. COMPILATION OF THE ANGLE FACTORS

In the following the analytical solutions for the four remaining angle factors will be given. In order to show the relationships of the angle factors for the different transitions, which are a consequence of the enclosure property, only a few abbreviations in the resultant expressions are used. The following nomenclature is employed. The height of the cylinders is denoted by H. The radii of the top and bottom surfaces are given by R_T and R_B, respectively. The radius of the inner cylinder is denoted by R_I and that of the outer cylinder by R_O. The configurations of the surfaces and their geometrical dimensions are shown below in detail for the individual radiant transfers. It is furthermore convenient to use dimensionless variables. Although other combinations are possible, the following variables appear to be advantageous:

$$X_T = \frac{R_T}{H}, \quad X_B = \frac{R_B}{H}, \quad X_O = \frac{R_O}{H}, \quad X_I = \frac{R_I}{H}.$$

Numerically evaluating the inverse trigonometric functions, which appear in the following expressions for the angle factors, the principal values have to be taken.

Transfer outer → outer

The transfer outer → outer describes the radiation transport from the interior surface of the outer cylinder to itself. The situation is illustrated in Fig. 4. The expression derived for the corresponding angle factor is

$$t_{O\to O} = \frac{1}{\pi X_O}\left[\pi(X_O - X_I) + \arccos\frac{X_I}{X_O} - \sqrt{(1+4X_O^2)}\arctan\frac{\sqrt{([1+4X_O^2][X_O^2 - X_I^2])}}{X_I} + 2X_I\arctan\left(2\sqrt{(X_O^2 - X_I^2)}\right)\right]. \quad (15)$$

Because of its compact form the above equation is clearer than that normally given in angle factor catalogues [1, 7, 8] and accordingly has the advantage that

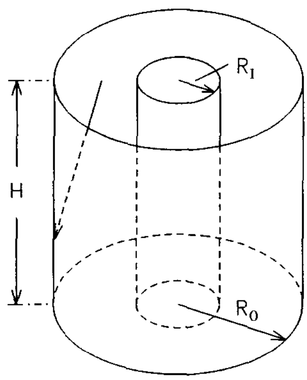

FIG. 4. Illustration of the transfer outer → outer.

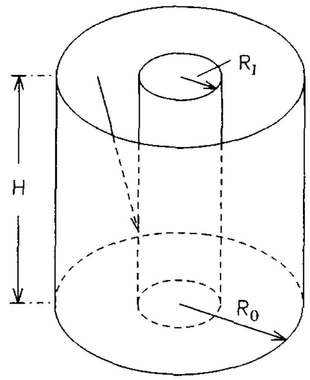

FIG. 5. Illustration of the transfer outer → inner.

it can be handled more easily in practical applications. If the radius of the outer cylinder is equal to the radius of the inner cylinder, i.e. for $X_O = X_I$, the angle factor $t_{O\to O}$ vanishes. If the radius of the inner cylinder is zero, i.e. if $X_I = 0$, it follows that

$$t_{O\to O} = \frac{1}{2X_O}\left[1 + 2X_O - \sqrt{(1+4X_O^2)}\right]. \quad (16)$$

Transfer outer → inner

The configuration of surfaces for the transfer outer → inner is shown in Fig. 5. The radiation travels from the interior surface of the outer cylinder to the exterior surface of the inner cylinder. An analytical solution for the corresponding angle factor is available in the literature [1, 7, 8]. A new expression with a consistent nomenclature has been derived in this study and is given here for completeness. The result is

$$t_{O\to I} = \frac{1}{\pi X_O}\left[\tfrac{1}{2}(X_O^2 - X_I^2 - 1)\arccos\frac{X_I}{X_O} + \pi X_I - \frac{\pi}{2}(X_O^2 - X_I^2) - 2X_I\arctan\sqrt{(X_O^2 - X_I^2)} + \sqrt{\{[1+(X_O+X_I)^2][1+(X_O-X_I)^2]\}} \times \arctan\sqrt{\left\{\frac{[1+(X_O+X_I)^2][X_O-X_I]}{[1+(X_O-X_I)^2][X_O+X_I]}\right\}}\right]. \quad (17)$$

For $X_O = X_I$ the angle factor $t_{O\to I}$ becomes unity and for $X_I = 0$ it vanishes.

Transition top → outer

Figure 6 illustrates the surface configuration for the transfer top → outer. The radiation passes from an annular disk at the top end of cylinder to the interior surface of the outer cylinder. An analytical solution for this transfer has not apparently been obtained previously. The resultant expression has the form

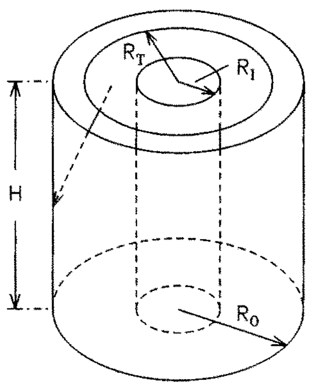

FIG. 6. Illustration of the transfer top → outer.

$$t_{T\to O} = \frac{1}{\pi(X_T^2-X_I^2)}\left[\tfrac{1}{2}(X_O^2-X_I^2)\left(\pi-\arccos\frac{X_I}{X_O}\right)\right.$$
$$-2X_I\{\arctan(\sqrt{(X_O^2-X_I^2)}+\sqrt{(X_T^2-X_I^2)})$$
$$-\arctan\sqrt{(X_O^2-X_I^2)}\}-\tfrac{1}{2}\arccos\frac{X_I}{X_T}$$
$$+\sqrt{\{[1+(X_O+X_T)^2][1+(X_O-X_T)^2]\}}$$
$$\times\arctan\sqrt{\left\{\frac{[1+(X_O+X_T)^2][Y^2-(X_O-X_T)^2]}{[1+(X_O-X_T)^2][(X_O+X_T)^2-Y^2]}\right\}}$$
$$-\sqrt{\{[1+(X_O+X_I)^2][1+(X_O-X_I)^2]\}}$$
$$\times\arctan\sqrt{\left\{\frac{[1+(X_O+X_I)^2][X_O-X_I]}{[1+(X_O-X_I)^2][X_O+X_I]}\right\}}$$
$$-(X_O^2-X_T^2)\arctan\left\{\frac{X_O+X_T}{X_O-X_T}\right.$$
$$\left.\left.\times\sqrt{\left(\frac{Y^2-(X_O-X_T)^2}{(X_O+X_T)^2-Y^2}\right)}\right\}\right] \quad (18)$$

where

$$Y=\sqrt{(X_O^2-X_I^2)}+\sqrt{(X_T^2-X_I^2)}. \quad (19)$$

If $X_I = 0$, the following expression is obtained:

$$t_{T\to O} = \frac{1}{2X_T^2}[\sqrt{([1+(X_T+X_O)^2][1+(X_T-X_O)^2])}$$
$$-1-X_O^2+X_T^2] \quad (20)$$

which has already been given with a different nomenclature in an earlier paper [7]. If the top circle radius is equal to the radius of the inner cylinder, i.e. if $X_T = X_I$, the expression in the square bracket of equation (18) vanishes.

Transition top → bottom

The configuration of surfaces for the transfer top → bottom is shown in Fig. 7. The radiation exits from an annular disk at the top end of cylinder and is intercepted by an annular disk at the bottom end of cylinder. The outer radii of the annular disks may be different. An analytical expression for the corresponding angle factor has not yet been given in the literature. The angle factor can be written as

$$t_{T\to B} = \frac{1}{\pi(X_T^2-X_I^2)}\left[\tfrac{1}{2}(X_T^2-X_I^2)\arccos\frac{X_I}{X_B}\right.$$
$$+\tfrac{1}{2}(X_B^2-X_I^2)\arccos\frac{X_I}{X_T}$$
$$+2X_I\{\arctan(\sqrt{(X_T^2-X_I^2)}+\sqrt{(X_B^2-X_I^2)})$$
$$-\arctan\sqrt{(X_T^2-X_I^2)}-\arctan\sqrt{(X_B^2-X_I^2)}\}$$
$$-\sqrt{\{[1+(X_T+X_B)^2][1+(X_T-X_B)^2]\}}$$
$$\times\arctan\sqrt{\left\{\frac{[1+(X_T+X_B)^2[Y^2-(X_T-X_B)^2]}{[1+(X_T-X_B)^2][(X_T+X_B)^2-Y^2]}\right\}}$$
$$+\sqrt{\{[1+(X_T+X_I)^2][1+(X_T-X_I)^2]\}}$$
$$\times\arctan\sqrt{\left\{\frac{[1+(X_T+X_I)^2][X_T-X_I]}{[1+(X_T-X_I)^2][X_T+X_I]}\right\}}$$
$$+\sqrt{\{[1+(X_B+X_I)^2][1+(X_B-X_I)^2]\}}$$
$$\left.\times\arctan\sqrt{\left\{\frac{[1+(X_B+X_I)^2][X_B-X_I]}{[1+(X_B-X_I)^2][X_B+X_I]}\right\}}\right] \quad (21)$$

where

$$Y=\sqrt{(X_T^2-X_I^2)}+\sqrt{(X_B^2-X_I^2)}. \quad (22)$$

If the bottom circle radius is equal to the radius of the inner cylinder, i.e. if $X_B = X_I$, the angle factor $t_{T\to B}$ vanishes. For $X_I = 0$, equation (21) goes over into the equation

$$t_{T\to B} = \frac{1}{2X_T^2}[1+X_T^2+X_B^2$$
$$-\sqrt{\{[1+(X_T+X_B)^2][1+(X_T-X_B)^2]\}}] \quad (23)$$

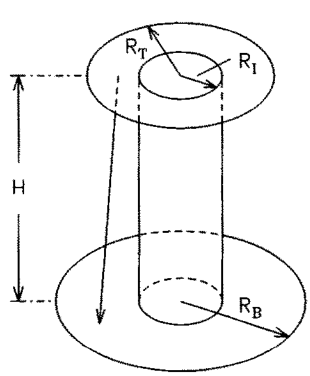

FIG. 7. Illustration of the transfer top → bottom.

which has already been tabulated in a different form [7].

Some of the algebraic expressions which appear in the above equations have simple geometric counterparts. This can be shown, for example, if the projection of the cylinders into the xy-plane is considered and a tangent on the inner cylinder surface is drawn. For the transition top → outer the quantity Y in equation (19) is then the distance from the intersection of the tangent with the circle of the top annular disk to the intersection of the tangent with the outer cylinder surface. The same consideration can be made for the transition top → bottom. In order to write the resulting equations in an even more compact form, it will be convenient to employ such expressions as abbreviations which have a geometrical meaning.

4. CONCLUSIONS

The angle-factor concept is used to calculate the heat or neutral particle transport in systems containing voids embedded in material regions. In this study analytical angle factors for the radiant transfers between the surface elements of two concentric cylinders are derived. By this the use of the angle factors in practical applications is simplified and the computational time necessary to evaluate the angle factors numerically can be reduced considerably. This is of importance if configurations with a large number of area elements at the void surface are studied.

There are thirteen possible geometric transitions between the surface elements in the system considered. The angle factors for nine of these transitions can be determined by using the angle-factor algebra. The remaining four angles are evaluated analytically. This is accomplished by replacing the integration over one of the surfaces by an integration over the direction of the radiation emitted by that surface. Due to the enclosure property the angle factors for the different transitions are not independent of each other and their relationships can be shown if a consistent notation is used and only a few abbreviations are employed in the resultant expressions. Even in this case the equations can be brought into such a form that the expressions are not too lengthy. The angle factors thus obtained can be easily evaluated numerically for later use in heat transfer or neutral particle streaming calculations in r–z geometry.

REFERENCES

1. E. M. Sparrow and R. D. Cess, *Radiation Heat Transfer*. Brooks/Cole, Belmont, California (1966).
2. H. Vossebrecker, Two-dimensional shielding calculations using a combination of diffusion theory and line-of-sight method, *Atomkernenergie* (*ATKE*) **19,** 14–16 (1972).
3. B. A. Clark, The development and application of the discrete ordinates–transfer matrix hybrid method for deterministic streaming calculations, LA-9357-T, Los Alamos National Laboratory (1982).
4. H. Brockmann, Improved treatment of two-dimensional neutral particle transport through voids within the discrete ordinates method by use of generalized view factors. In *Deterministic Methods in Radiation Transport* (Edited by A. F. Rice and R. W. Roussin), ORNL/RSIC 54, Oak Ridge National Laboratory (1992).
5. K. D. Lathrop, Discrete ordinates methods for the numerical solution of the transport equation, *Reactor Technol.* **15,** 107–135 (1972).
6. VDI-Wärmeatlas, Berechnungsblätter für den Wärmeübergang (6th Edn), Chap. Ka. VDI, Düsseldorf (1991).
7. H. Leuenberger and R. A. Person, Compilation of radiation shape factors for cylindrical assemblies, ASME Annual Meeting, New York, ASME Paper No. 56-A-144 (1956).
8. J. R. Howell, *A Catalog of Radiation Configuration Factors*. McGraw-Hill, New York (1982).